The Time Has Come: Interest Rate Modeling For Liability Cash Flow Analysis

Intuitive Analytics

Third Edition, October 2008
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Introduction to and Motive for Interest Rate Models

Historically, interest rate models have largely been the domain of finance professionals with advanced quantitative degrees called "quants." The mission of these financial engineers is often solely to accurately value interest rate sensitive instruments for the purposes of trading or risk management, usually within financial services firms. However, a frequently overlooked yet straightforward application of interest rate models is equally beneficial to those managing interest rate sensitive liabilities in the modern corporation.

Why interest rate models are not used more frequently is an interesting question with no clear good answers, though associated with some noteworthy observations. First, these models are largely misunderstood, described in antiseptic, stark, mathematical terms, and as such are thought of as inaccessible to most and generally too complex to use. In fact, those who understand these types of tools often possess an inherent bias against making them better known because that would only dilute the value of their hard won knowledge. Second, a model's useful implementation often requires greater computing power and flexibility than the ubiquitous spreadsheet. Yet sadly for many finance professionals, the spreadsheet is the only flexible computational tool available. The third reason is more subtle and leads to an overlooked yet incredibly important simplifying assumption. This is explored in more detail below.

For many corporations, as the saying goes, cash is king. Add current GAAP accounting standards and the result is that many liability managers are focused on debt capital cost and related interest expense as their biggest capital markets exposure. The change in present value of the liability or debt portfolio is a second order consideration[1]. As a result, fixed rate debt is often considered riskless (fixed nominal cost) and floating rate debt is the stuff of sleepless nights. In other words, there is a fundamentally different perspective on losing cash out the door due to an unexpected increase in the cost of variable rate bonds versus the opportunity cost of having issued fixed rate debt in a high rate environment, and subsequently experiencing a decline in long term yields. We could long debate the theoretical "correctness" of viewing risk in this fashion, but it is simply a current fact of life for many corporations. Perhaps a reasonable justification is that the former risk, higher floating rate debt cost, can lead ultimately to a problem as serious as default. The other event, an opportunity loss associated with fixed rate debt value, would not.

Most quants are trained to think purely in present value terms[2]: the value of floating rate debt, no matter the interest rate environment, is always at

[1] RiskMetrics Group dubs those places where mark to market risk is the primary concern as "financial environments". This is contrasted with what they call "corporate environments" wherein earnings and cashflow are primary considerations.

[2] "I think in purely present value terms…" *Fisher Black in an Augutst, 1995 email to the author*

or near par. If there's no price movement in the instrument than it must be risk free. They would counter that it's the fixed rate bonds with volatile prices that are the ones that need the attention. Their perspective is understandable given their jobs are usually associated with managing daily mark to market valuation of financial portfolios. The volatility of that daily mark is the risk they manage and in that world, a floating rate instrument with a par price **would** be riskless.

The conclusion we draw is that few quants spend time looking at the problem of floating or short term interest rate risk because they find the perspective of the liability manager slightly irrational in the financial-economic sense. It's a problem unlikely to even cross a quant's desk.

Completing the round trip, if we acknowledge that short term rates drive legitimate risks, we enjoy the great luxury of focusing only on a short term interest rate model; the entire term structure of yields is not a concern, unless it somehow drives cashflow[3]. We also gleefully aren't required to calibrate our model to the market prices of bonds, swaps, options, or other instruments – a challenging task even under the best of circumstances that usually does require an advanced degree in a quantitative discipline.

[3]The relatively recent popularity of Constant Maturity Swaps obviously requires a model capturing term structure dynamics in order to capture the cash flow implications of such structures

This booklet and the companion Excel® workbook introduce a general class of short term interest rate model and explain the intuitive meaning of its parameters. This may be very new material to some but newer still may be its applicability to "real world" problems. If unfamiliar, the time spent absorbing these contents will provide a fresh context for evaluating capital market decisions and may provide a consistent, ongoing framework for assessing interest rate risk where no framework existed before.

Interest Rate Modeling

The purpose of interest rate models is to help us explain and understand interest rate behavior. Whether the specific application is forecasting, valuation, or risk management, interest rate models ultimately must capture characteristics of interest rates that are either expected based upon some view or implied by market prices. For a general introduction to continuous time finance and the topic of modeling market variables in finance, please see Appendix A.

In the end, our objective is not complicated. Where in the past those dealing with interest rate risk might use an estimate or forecast of rates to analyze an expectation of interest expense, we now wish to create a complete distribution for interest rates and quantify those risks. An entire distribution for rates can meaningfully inform our understanding of how bad things might get if the forecast were wrong. In fact, one interpretation of the width of these distributions is as a level of conviction about the quality of a forecast. Of one thing we can be certain, markets will change and these expectations will be adjusted right along with them.

As discussed in the introduction, we will focus our attention on short term interest rates as they are the basis for calculating cash flows. After picking a starting rate (99% of the time the current rate being modeled), there are four parameters that need to be specified for the model to function:

Average Rate, m – The average rate to which interest rates will trend in the model. The farther away a rate in the model is from *m*, the stronger the attraction to *m* will be.

Reversion Speed, s – This parameter gives us the rate of "pull" that rates experience towards the average rate, *m* (above), over the analytic time horizon.

Volatility, σ – This term, commonly known as "vol", allows us to control the rate of variability or dispersion of rates in the model. This is also the spot where we comply with the law that says you can't write a paper about interest rate models without at least one Greek letter; our humble lower case sigma, σ, is it.

Rate Exponential, p – This term tells us how much volatility changes with the level of rates i.e. if short term rates are hovering at 12%, we probably expect them to move around, in basis points, far more than if rates are 1%. This parameter tells us how much.

Each of these above parameters enters the model in a specific way in order to perform their respective tasks. Using the symbols above, the general short term interest rate model, whose terms will be explored in detail over the next few sections, takes the following form[1]:

$$dr_t = s(m - r)dt + \sigma r^p dZ$$

Although this may appear imposing, this equation has a straightforward, intuitive meaning. It says that a small change in interest rates (*dr*) is explained by two factors, one is relative to time (*dt*), called a "drift" term. The second induces some randomness into the picture and as a result is called a volatility term. The volatility term includes an increment of a random walk, *dZ*. We will put both the drift and volatility terms under the microscope in the sections ahead, but before we get too far, in order to implement a solution to the equation above and actually create interest rates in our model, we need to go from continuous time, to what is commonly called "discrete time."

Largely in the interest of computational ease and in order to cover long time horizons, we will only concern ourselves with certain specified increments in time, perhaps every month, quarter or even every year. Therefore, we have to convert our "continuous time" model above into an approximation that will work over specified time intervals. Luckily, our new approximation is nearly identical to the model above, with the dt replaced by a Δt.

In order to calculate a rate at time t, we have the following strategy,

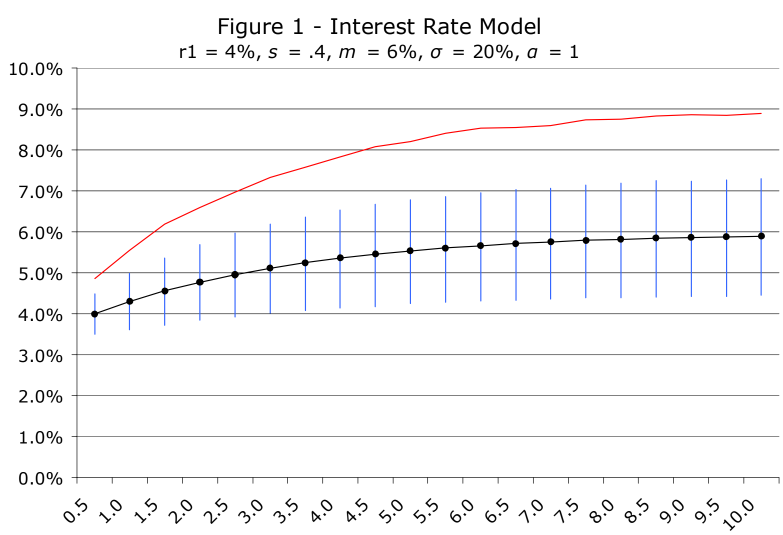
$$r_t = r_{t-1} + \Delta r_t$$

where

$$\Delta r_t = s(m - r_{t-1})\Delta t + \sigma r_{t-1}^{p}\sqrt{\Delta t}z_j$$

These two equations together describe a flexible numeric process for computing interest rate distributions over any time horizon desired. You'll notice we've changed dZ into $\sqrt{\Delta t}z_j$ where z_j is a random sample from a chosen probability distribution. The most commonly used distribution of course is the familiar, bell shaped, normal distribution. It is the random sampling of the z_j in the model that is commonly called a Monte Carlo method.

Figure 1 below represents a solution to the above equation over a ten year period with the starting rate equal to 4%, reversion speed s = .4, average rate m = 6%, time step Δt = .5 (semiannual), volatility σ = 20%, and rate exponent p = 1.



This figure was created by generating 10,000 realizations of the model above using a normal distribution for each sample, z_j. The black circles connected by the black line represent the average rate at each of the 20 semiannual time steps. The blue line above and below represent plus or minus one standard deviation for rates in the model; that is, roughly two thirds of the time rates fall within the constraints of the blue line. The red line above reflects the rate at the 95% highest place in the distribution, at each time step.

We've already reached our objective as the model above represents an entire distribution of rates for each semiannual period. After a little bit of heavy lifting, it's all down hill from here. Graphics highlighting various aspects of changes to our four model inputs will provide much of the insight going forward. The drift and volatility terms that make our model go – what they are and how they work, are explored in detail in the next two sections.

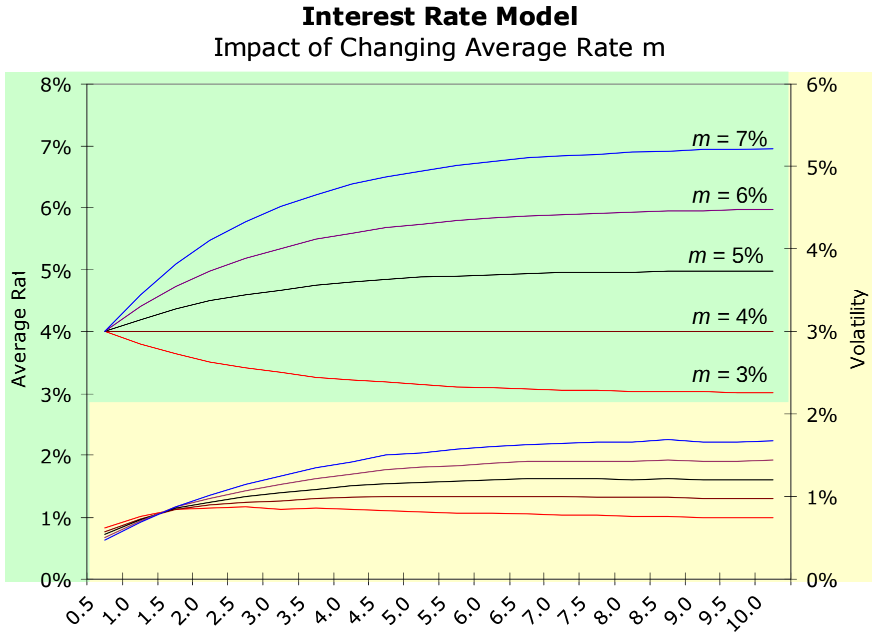
The Drift Term – How We EXPECT Things to Change

One observed characteristic of interest rates that a model should accommodate is that rates aren't allowed to vary "too much." That is, rates should not become negative nor should they assume implausibly large values. This feature can be incorporated in interest rate models by either explicitly creating a volatility term in the model that decreases with time, or by employing a drift term that tends to "pull" rates towards a long term average (m in our case). The general model in this paper uses the latter technique employing a mean reverting drift. Looking at the drift term alone from the equation above,

$$s(m - r_{t-1})\Delta t ,$$

where m is the mean rate towards which rates will pull in the model (6% in Figure 1), s is the speed that rates will tend to that average (0.4 in Figure 1), and r_{t-1} is simply the rate from the prior period along each simulated path.

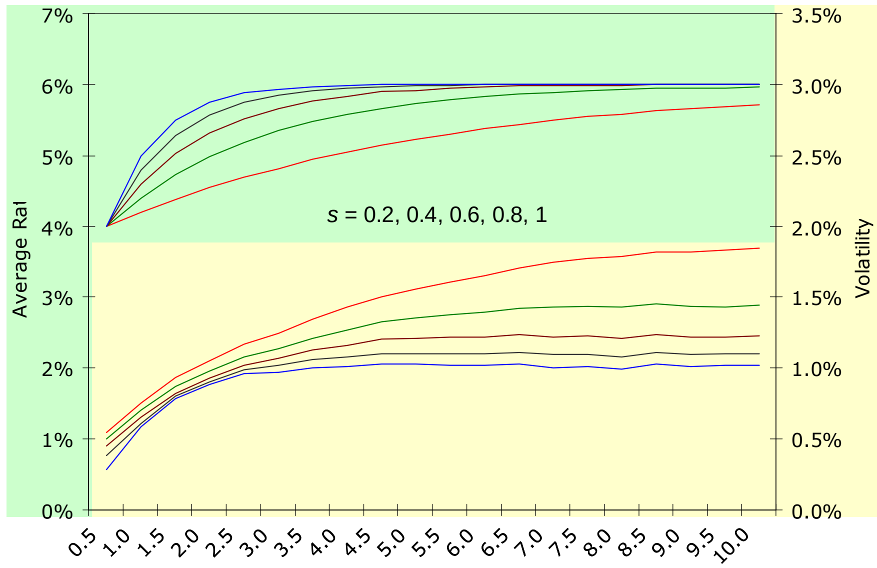
The chart below is created by running the model five separate times. Each of the five top lines below show the impact of changing the mean rate, m, in the model. Table 1 at the end of this booklet shows the data for this graph.



The graph above shows the impact of changing the average rate, *m*, by 1% increments from 3% to 7% on the mean rate path and volatility in the model. The initial rate in all cases is 4%. The top five lines associate with the left vertical axis and show the average rate path for each of the five model runs. The lower five lines are graphed relative to the right vertical axis and are colored to match the expected line with which they're related above. Notice that since the volatility term is scaled by rates in the model (i.e. $p = 1$), the higher the value for *m*, the higher the resulting rate volatility in the bottom lines.

The graph below shows the impact of changing the mean reversion speed, *s*, in the model; *s* is adjusted from 0.2 to 1 in increments of 0.2. The top five lines show the average rate path and the bottom five, with yellow backdrop, show the associated rate volatility over the 10 year period.

Interest Rate Model
Impact of Changing Mean Reversion Speed *s*



Notice that as *s* increases, the more quickly the average rate gets to *m*, which in this model for all five cases is 6%. You also can see that as *s* increases and the attraction to the mean rate, *m*, increases, overall volatility in the model deceases. The more extreme rates in the simulation are marshaled in to a greater degree with higher mean reversion.

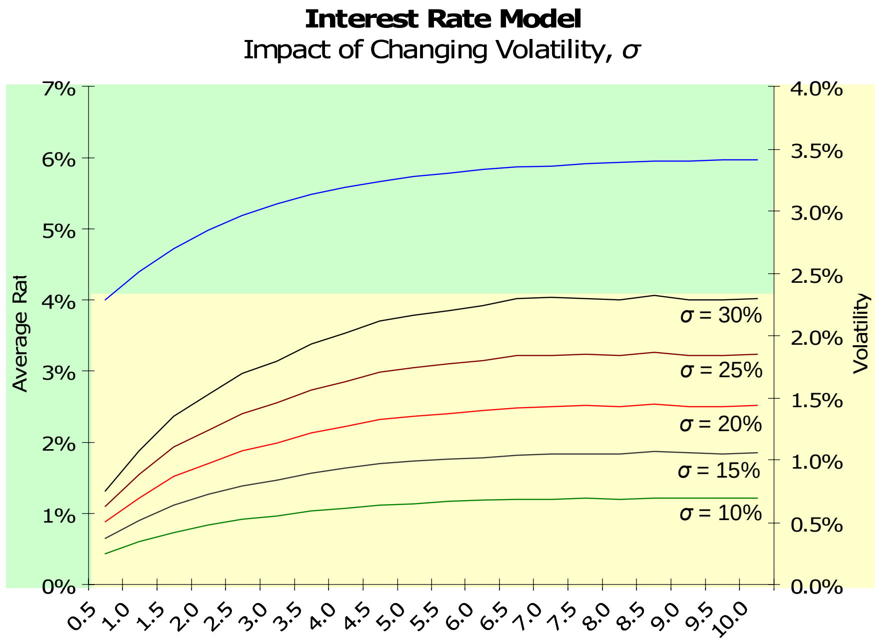
Volatility's Dance

The second term beginning with σ is the volatility term,

$$\sigma r_{t-1}^{p}\sqrt{\Delta t}\,z_j$$

The σ and its implication for rates are very much dependent upon the exponent for r_{t-1}, *p*. This *p* term determines whether the volatility of changes in rates is normally distributed ($p = 0$) or lognormally distributed ($p = 1$) or is some other value that better fits the view or modeling purposes of the user.

For each of the five model calculations below, *p* is equal to 1. A value of 1 is probably most commonly used by practitioners due to its consistency with the market convention for caps and vanilla swaptions, and the attractive feature of precluding negative interest rates.



In the graph above, the blue line indicates the average rate path for all five calculations; they are identical in each case. We have not changed anything in the drift term so there is no reason it should vary. What does change is the rate's volatility around the mean shown in the five lines in the bottom (yellow) section of the graph.

History's Lessons

There is no shortage of research investigating the nature of short term rates and in particular, how they can and should be accurately modeled. Unfortunately, much of this research is either inconclusive or highly dependent upon the historical period examined. For example, an article[4] in the *Financial Analysts Journal* summarizes some results:

> *Short rate fact #1.* The short-rate series is a "persistent" time series; that is, it spends long, consecutive periods above and below the [long run] mean.
> *Short-rate fact #2.* In the 1979-82 period, the average level of and volatility of the short rate was substantially higher than for other years in the 1971-2000 period.
> *Short-rate fact #3.* The volatility of the short rate-level appears to be both time varying and persistent."

Unfortunately, none of these "facts" make it easier to reliably say anything of great value about how to estimate parameters in our model. Fact #1 is likely dependent upon business cycles which are notoriously difficult to forecast; Fact #2 reflects the monetary experiment of that period with the Fed targeting the money supply instead of rates and letting rates, aka the "cost of money", set at market clearing levels; Fact #3 is influenced strongly by the level of rates and activity, or lack

[4] Chapman, David A. and N. Pearson. "Recent Advances in Estimating Term-Structure Models." Financial Analyst Journal, July/August 2001, 77-95.

thereof, by the Federal Reserve.

What about mean reversion, *s*? Certain statistical methods are available for estimating this parameter[5]. Using long term data (back to at least 1980), statistical methods put a best fit for *s* in the range of 0.05 to 0.3. Looking more closely at the data, it appears that modeling mean reversion in the manner that best fits history would indicate close to 0 mean reversion when rates are in a "normal" range, and then high mean reversion when rates are at their extremes. Fitting this to a model exactly is extremely difficult largely because of the scarcity of data at the extremes. This occurs almost by definition, of course; if we had a bounty of data at the extremes then they would no longer be extremes.

[5] Aït-Sahalia, Yacine. "Transition Densities for Interest Rate and Other Nonlinear Diffusions." *The Journal of Finance* Vol. LIV. No. 4, August 1999, 1361-1395.

The volatility parameters, both σ and p, are highly dependent on the data analyzed and in particular the 1979-81 Fed period. If this period is included, a number of researchers have found a best fit with p between 1.5 and 2. If this period is excluded, the best answer tends to fall in the vicinity of 1.

All Together

We have reviewed the components of a flexible and powerful interest rate model and seen how the four main parameters affect simulated interest rate behavior. However, for those who want to internalize this material, the companion excel workbook, "SmartModels Rate Primer.xls" provides the ability to change these parameters and immediately explore the results of those adjustments.

An interest rate model in isolation offers little utility, however. It is only a first step, albeit an important one, in ultimately understanding a capital structure's exposure to interest rate variation. The nature of aggregate financial risk is understood by exploring how different market elements tend to co-move or correlate: LIBOR and BMA for instance. To do this, fortunately we can use the exact same model introduced above. However, instead of modeling an interest rate, we'll model the relationship between one rate and another, often called the "basis." However, we'll save that topic for its own discussion.

Appendix A – Friendly Introduction to Continuous Time Finance

Modern day interest rate models are a subset of a field called continuous time finance, so named because the mechanics of these financial models require an assumption that prices and markets are in continuous, unbroken motion (perhaps the most famous violation of that assumption was the crash of 1987 where these types of models failed spectacularly). The modern roots of continuous time finance trace to the celebrated Black and Scholes' option pricing paper and Robert Merton's seminal work on the subject in the early 1970s (see *Additional Reading*).

Although many variations exist, in general, this type of modeling begins with a diffusion equation,

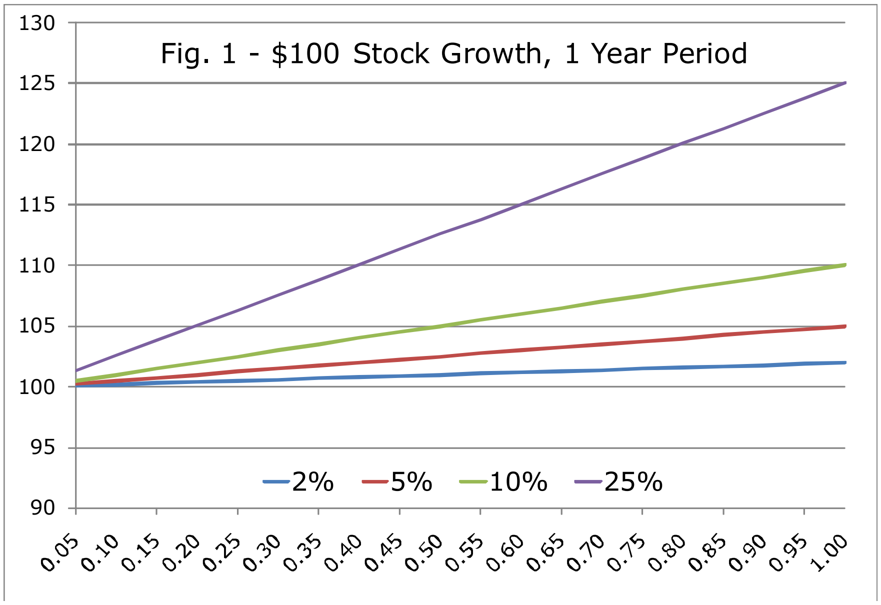
$$dX_t = a(X,t)dt + b(X,t)dZ$$

Before the equation-phobic reader takes flight, please stay in the game. There are some very intuitive, pictographic ways to achieve a fundamental understanding of what this equation means and once attained, will bear great dividends.

This equation describes a formula for the infinitesimally small change in a variable, X_t. This is a *very* general formulation. Our X_t could be anything – a stock price, an interest rate, the cost of gold, a currency rate, the price of tea in Turkey or coffee in Brazil, even the temperature in downtown Manhattan. It is simply a model to describe how some amount of something changes through time.

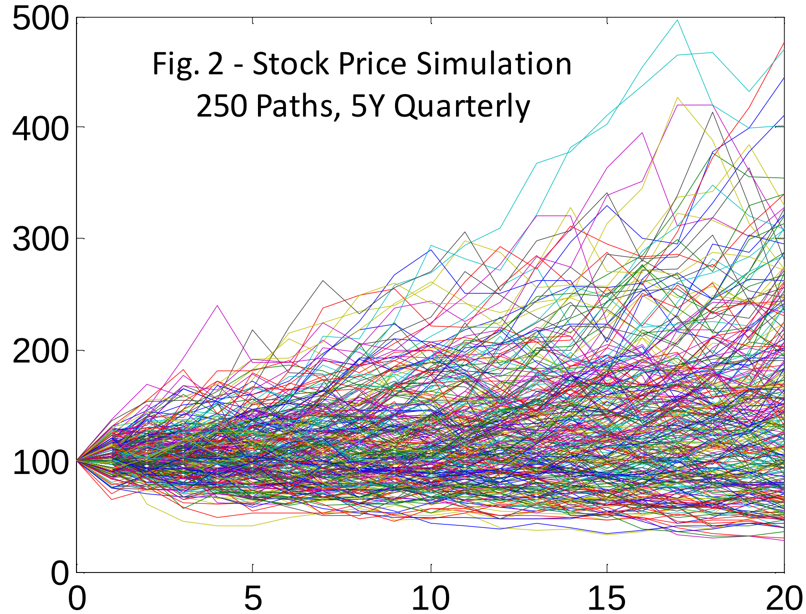
The first term, a, is often called the "drift" term as it describes the average movement in X_t per unit of time, dt. In the Black-Scholes-Merton world, a is simply a constant representing the expected growth rate of X_t, where X_t represents the price of a stock. For instance, a may be .1, representing a 10% growth rate annually.

Ignoring the b term for the moment and assuming our X_0 is 100, Fig 1 shows the growth of X_t over a one year period assuming different constant growth rate levels for a.



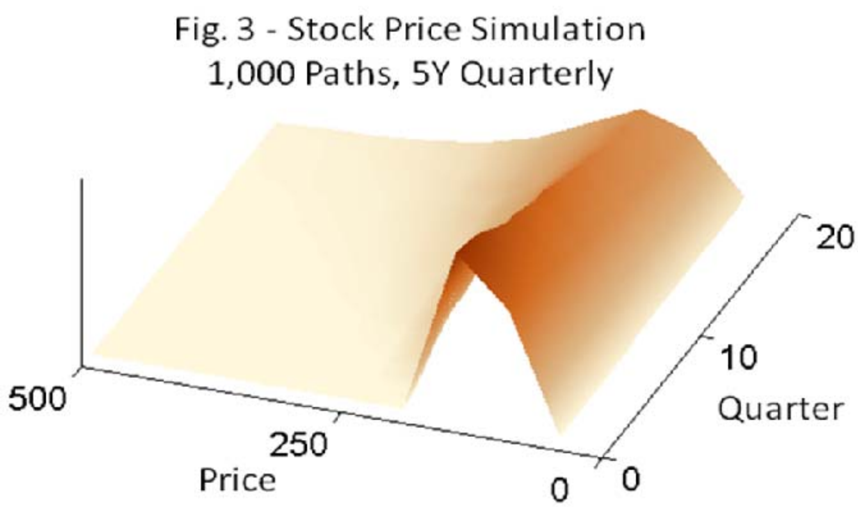
The b term above is often called a noise or "volatility" term as it imparts randomness into the process. In general, the higher value the b function takes, the greater X_t will disperse through time, moving in a more random fashion. The dZ term is the increment of a Brownian motion. Brownian motion is commonly referred to as a "random walk" and is used in a variety of settings to model how physical (or financial) phenomena with random characteristics change over time.

We now take our simple stock model above and add volatility to it to generate the simulation in Figure 2.

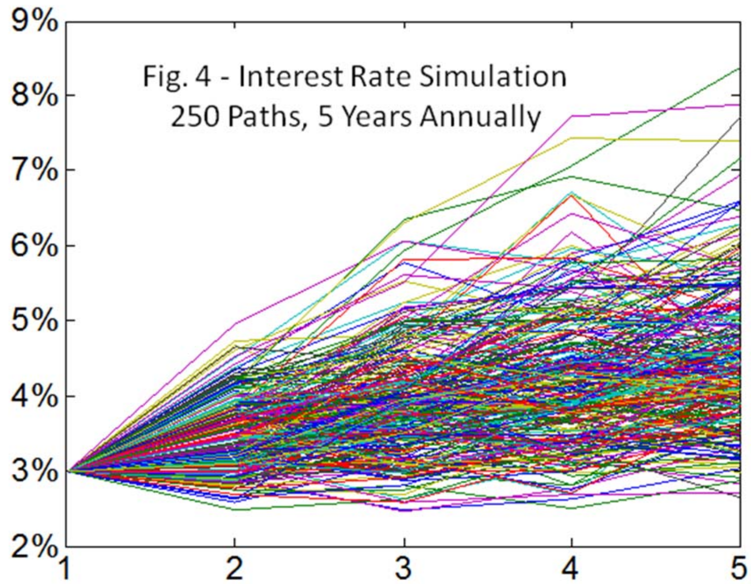


In this case, we've simulated a stock with a \$100 initial price quarterly over five years (20 quarters). The stock's expected growth rate annually (*a*) is 10% with a volatility 25%. This 25% is scaled by the stock price itself. Specifically, this means that the higher the price of the stock within the simulation, the greater the volatility in absolute dollar terms i.e. the stock will tend to be more volatile in price terms at \$200 than it is at \$100. This is a property of stock price volatility that roughly holds in the real world.

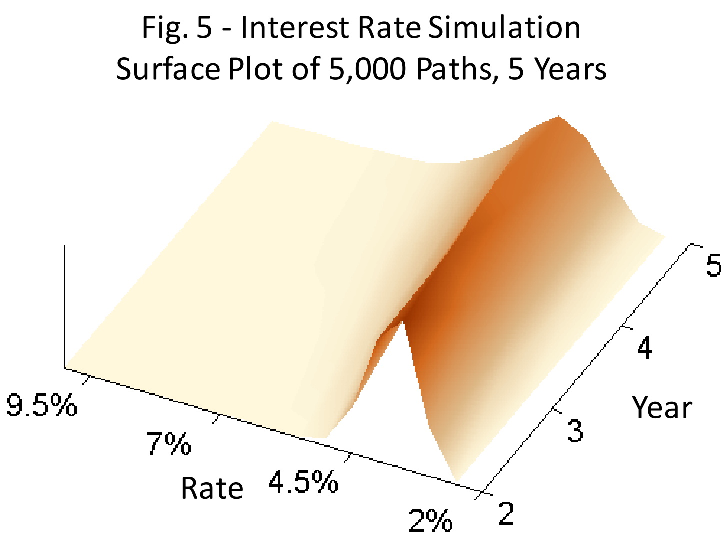
In Figure 3 below, we show the exact same model for our stock above, however, we show it using a surface plot of the histogram of the stock price stock through time. Notice that as time progresses through the five years, the "width" of the distribution for the stock price increases. This is standard behavior of the model as our uncertainty about the future increases with (the square root of) time.



In Figure 4 on the following page is the same basic model however now we use a mean reverting drift as described in the body of this paper. This is an annual simulation of interest rates over five years. Notice that the simulations, on average, are moving up over the time period of the analysis. This is a result of a starting interest rate of 3% moving towards an average interest rate of 5% and a positive reversion speed.



Our last graph shows the same information above but again in a surface plot. This plot shows how the distribution of interest rates in the model changes over time.



We have broken up our continuous time model for how almost anything with a random nature can change (or diffuse) through time into a discrete time allegory of the same animal. Each time step we employ the same recipe for generating a new state of the world.

Appendix B – Tables

Table 1
Interest Rate Model Sensitivity to Average Rate Changes
r1 = 4%, s = .4, σ = 25%, α = 1

Semi Period	Year	m = 3%		m = 4%		m = 5%		m = 6%		m = 7%	
1	0.5	4.00%	0.61%	4.00%	0.57%	4.00%	0.54%	4.00%	0.50%	4.00%	0.47%
2	1.0	3.80%	0.75%	4.00%	0.73%	4.20%	0.72%	4.40%	0.70%	4.60%	0.68%
3	1.5	3.64%	0.84%	4.00%	0.85%	4.36%	0.86%	4.72%	0.87%	5.08%	0.88%
4	2.0	3.51%	0.86%	4.00%	0.89%	4.49%	0.93%	4.98%	0.97%	5.46%	1.02%
5	2.5	3.41%	0.87%	4.00%	0.93%	4.59%	1.00%	5.18%	1.08%	5.77%	1.15%
6	3.0	3.33%	0.85%	4.00%	0.94%	4.67%	1.04%	5.34%	1.14%	6.01%	1.24%
7	3.5	3.26%	0.85%	4.00%	0.97%	4.74%	1.09%	5.48%	1.21%	6.21%	1.34%
8	4.0	3.21%	0.84%	4.00%	0.98%	4.79%	1.12%	5.58%	1.27%	6.37%	1.42%
9	4.5	3.17%	0.84%	4.00%	1.00%	4.84%	1.16%	5.67%	1.33%	6.50%	1.49%
10	5.0	3.14%	0.82%	4.00%	0.99%	4.87%	1.17%	5.73%	1.35%	6.60%	1.53%
11	5.5	3.11%	0.80%	4.00%	0.99%	4.89%	1.18%	5.78%	1.38%	6.68%	1.57%
12	6.0	3.09%	0.79%	4.00%	0.99%	4.91%	1.19%	5.83%	1.40%	6.74%	1.60%
13	6.5	3.07%	0.79%	4.00%	1.00%	4.93%	1.21%	5.86%	1.42%	6.79%	1.64%
14	7.0	3.05%	0.78%	4.00%	0.99%	4.94%	1.21%	5.89%	1.43%	6.83%	1.64%
15	7.5	3.04%	0.77%	4.00%	0.99%	4.95%	1.21%	5.91%	1.43%	6.87%	1.66%
16	8.0	3.03%	0.76%	4.00%	0.98%	4.96%	1.20%	5.93%	1.43%	6.89%	1.65%
17	8.5	3.03%	0.76%	4.00%	0.99%	4.97%	1.22%	5.94%	1.45%	6.92%	1.68%
18	9.0	3.02%	0.74%	4.00%	0.97%	4.97%	1.20%	5.95%	1.43%	6.93%	1.66%
19	9.5	3.01%	0.74%	3.99%	0.97%	4.97%	1.20%	5.95%	1.43%	6.93%	1.66%
20	10.0	3.01%	0.74%	3.99%	0.97%	4.98%	1.21%	5.96%	1.44%	6.95%	1.68%

Table 2
Interest Rate Model Sensitivity to Reversion Speed Changes
r1 = 4%, m = 5%, σ = 25%, α = 1

Semi Period	Year	s = .2		s = .4		s = .6		s = .8		s = 1	
1	0.5	4.00%	0.54%	4.00%	0.50%	4.00%	0.45%	4.00%	0.38%	4.00%	0.29%
2	1.0	4.20%	0.75%	4.40%	0.70%	4.60%	0.65%	4.80%	0.61%	5.00%	0.58%
3	1.5	4.38%	0.93%	4.72%	0.87%	5.02%	0.82%	5.28%	0.80%	5.50%	0.79%
4	2.0	4.54%	1.05%	4.98%	0.97%	5.31%	0.93%	5.57%	0.90%	5.75%	0.89%
5	2.5	4.69%	1.17%	5.18%	1.08%	5.52%	1.02%	5.74%	0.99%	5.87%	0.96%
6	3.0	4.82%	1.25%	5.34%	1.14%	5.66%	1.07%	5.84%	1.02%	5.94%	0.97%
7	3.5	4.94%	1.35%	5.48%	1.21%	5.77%	1.13%	5.91%	1.06%	5.97%	1.00%
8	4.0	5.05%	1.43%	5.58%	1.27%	5.84%	1.16%	5.95%	1.08%	5.99%	1.01%
9	4.5	5.15%	1.50%	5.67%	1.33%	5.89%	1.20%	5.97%	1.10%	6.00%	1.03%
10	5.0	5.23%	1.55%	5.73%	1.35%	5.92%	1.21%	5.98%	1.10%	6.00%	1.02%
11	5.5	5.30%	1.60%	5.78%	1.38%	5.94%	1.22%	5.99%	1.10%	6.00%	1.02%
12	6.0	5.37%	1.66%	5.83%	1.40%	5.96%	1.22%	5.99%	1.10%	6.00%	1.02%
13	6.5	5.44%	1.70%	5.86%	1.42%	5.97%	1.24%	6.00%	1.11%	6.00%	1.03%
14	7.0	5.49%	1.74%	5.89%	1.43%	5.98%	1.22%	6.00%	1.09%	6.00%	1.01%
15	7.5	5.54%	1.77%	5.91%	1.43%	5.98%	1.23%	6.00%	1.10%	6.00%	1.01%
16	8.0	5.59%	1.79%	5.93%	1.43%	5.99%	1.21%	6.00%	1.08%	6.00%	1.00%
17	8.5	5.63%	1.82%	5.94%	1.45%	6.00%	1.24%	6.00%	1.11%	6.00%	1.03%
18	9.0	5.66%	1.82%	5.95%	1.43%	5.99%	1.22%	6.00%	1.09%	6.00%	1.01%
19	9.5	5.69%	1.83%	5.95%	1.43%	5.99%	1.22%	6.00%	1.10%	6.00%	1.02%
20	10.0	5.72%	1.84%	5.96%	1.44%	5.99%	1.23%	6.00%	1.10%	6.00%	1.02%

Table 3
Interest Rate Model Sensitivity to Volatility Changes
r1 = 4%, m = 6%, s =.4, α = 1

Semi Period	Year	σ = 10%		σ = 15%		σ = 20%		σ = 25%		σ = 30%	
1	0.5	4.00%	0.25%	4.00%	0.38%	4.00%	0.50%	4.00%	0.63%	4.00%	0.76%
2	1.0	4.40%	0.34%	4.40%	0.52%	4.40%	0.70%	4.40%	0.88%	4.40%	1.08%
3	1.5	4.72%	0.42%	4.72%	0.64%	4.72%	0.87%	4.72%	1.10%	4.72%	1.35%
4	2.0	4.98%	0.48%	4.98%	0.72%	4.98%	0.97%	4.98%	1.24%	4.98%	1.53%
5	2.5	5.18%	0.52%	5.18%	0.79%	5.18%	1.08%	5.18%	1.37%	5.18%	1.69%
6	3.0	5.34%	0.55%	5.34%	0.84%	5.34%	1.14%	5.34%	1.45%	5.34%	1.79%
7	3.5	5.48%	0.59%	5.48%	0.89%	5.48%	1.21%	5.48%	1.56%	5.48%	1.93%
8	4.0	5.58%	0.61%	5.58%	0.93%	5.58%	1.27%	5.58%	1.63%	5.59%	2.02%
9	4.5	5.67%	0.64%	5.67%	0.97%	5.67%	1.33%	5.67%	1.70%	5.68%	2.12%
10	5.0	5.73%	0.65%	5.73%	0.99%	5.73%	1.35%	5.73%	1.74%	5.73%	2.16%
11	5.5	5.78%	0.66%	5.78%	1.01%	5.78%	1.38%	5.78%	1.77%	5.78%	2.20%
12	6.0	5.83%	0.67%	5.83%	1.02%	5.83%	1.40%	5.83%	1.80%	5.83%	2.24%
13	6.5	5.86%	0.68%	5.86%	1.04%	5.86%	1.42%	5.86%	1.84%	5.87%	2.30%
14	7.0	5.89%	0.68%	5.89%	1.04%	5.89%	1.43%	5.89%	1.84%	5.89%	2.31%
15	7.5	5.91%	0.69%	5.91%	1.05%	5.91%	1.43%	5.91%	1.84%	5.91%	2.29%
16	8.0	5.93%	0.69%	5.93%	1.05%	5.93%	1.43%	5.92%	1.84%	5.92%	2.29%
17	8.5	5.94%	0.70%	5.94%	1.06%	5.94%	1.45%	5.94%	1.86%	5.94%	2.32%
18	9.0	5.95%	0.69%	5.95%	1.05%	5.95%	1.43%	5.95%	1.84%	5.94%	2.29%
19	9.5	5.96%	0.69%	5.96%	1.05%	5.95%	1.43%	5.95%	1.84%	5.94%	2.28%
20	10.0	5.97%	0.70%	5.97%	1.06%	5.96%	1.44%	5.96%	1.85%	5.95%	2.29%

Additional Reading

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